Operator Decomposition of Graphs

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Abstract: *In this paper we introduce a new form of decomposition of graphs, the (P, Q)-decomposition. We first give an optimal algorithm for finding the 1-decomposition of a graph which is a special case of the (P, Q)-decomposition which was first introduced in [21]. We then examine the connections between the 1-decomposition and well known forms of decomposition of graphs, namely, modular and homogeneous decomposition. The characterization of graphs totally decomposable by 1-decomposition is also given. The last part of our paper is devoted to a generalization of the 1-decomposition. We first show that some basic properties of modular decomposition can be extended in a new form of decomposition of graphs that we called operator decomposition. We introduce the notion of a (P, Q)-module, where P and Q are hereditary graph-theoretic properties, the notion of a (P, Q)-split graph and the closed hereditary class (P, Q) of graphs (P and Q are closed under the operations of join of graphs and disjoint union of graphs, respectively). On this base, we construct a special case of the operator decomposition that is called (P, Q)-decomposition. Such decomposition is uniquely determined by an arbitrary minimal nontrivial (P, Q)-module in G. In particular, if G $\not\in$ (P, Q), then G has the unique canonical (P, Q)-decomposition.*

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1. Introduction

All graphs considered are finite, undirected, without loops and multiple edges. For all notions not defined here the reader is refered to [3]. The vertex and the edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively, while n denotes the cardinality of $V(G)$ and m the cardinality of $E(G)$. We write $u \sim v\,(u \not\sim v)$ if vertices u and v are adjacent (non-adjacent). For the subsets $U, W \subseteq V(G)$ the notation $U \sim W$ means that $u \sim w$ for all vertices $u \in U$ and $w \in W$, $U \not\sim W$ means that there are no adjacent vertices $u \in U$ and $w \in W$. To shorten notation, we write $u \sim W$ ($u \not\sim W$) instead of $\{u\} \sim W$ ($\{u\} \not\sim W$). The subgraph of G induced by a set $A \subseteq V(G)$ is denoted by $G[A]$. We write $\overline{G}$ for the complement graph of G. The neighborhood of a vertex v in the graph G is denoted by $N_G(v)$ (or $N(v)$), $\overline{N}_G(v) = V(G) \setminus v \setminus N_G(v)$.

One type of graph decomposition based on the well-known notion of split graphs is investigated. A triad $T = (G, A, B)$, where G is a graph and (A, B) is an ordered bipartition of $V(G)$ into a clique A and a stable set B, is considered as an operator acting on the set of graphs. An operator T acts on a graph H by formula:

$$TH = G \cup H \cup \{ax / a \in A, x \in V(H)\} \qquad (1)$$

(all edges of the complete bipartite graph with the parts A and $V(H)$ are added to the disjoint union $G \cup H$).

An isomorphism of triads is defined as an isomorphism of 2-colored graphs. Denote by Tr the set of triads distinguished up to isomorphism of triads. The action (1) induces the associative binary operation on Tr. So the set Tr becomes a semigroup of operators with the exact action on the set of graphs. The semigroup Tr was introduced in [21]. The following structure theorem of the decomposition was presented in the same paper.

A graph F is called decomposable if there exist a triad T and a graph H such that $F = TH$, otherwise F is indecomposable. The decomposition theorem asserts that every decomposable graph F can be uniquely represented in the form:

$$F = T_1 T_2 \ldots T_k F_0$$

Where T_i is indecomposable element of the semigroup Tr and F_0 is indecomposable graph. This theorem occurs to be useful instrument for the characterization and enumeration of several graph classes [19, 22]. On the base of the theorem, the Kelly-Ulam reconstruction conjecture was proved for the class of decomposable graphs. A criterium of decomposability of graphs was presented in [23]. In the same paper on the base of the decomposition theorem an exhaustive description of unigraphs was obtained. (A graph is called a unigraph if it is determined uniquely up to isomorphism by its degree sequence). Namely, it was proved that a graph is a unigraph if and only if all its indecomposable components are unigraphs, and the catalogue of indecomposable unigraphs was given.

In this paper, a decomposition theory is developed. In section 2, we present the 1-decomposition which